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ON THE DISCRETIZATION OF AN ELASTIC ROD WITH DISTRIBUTED SLIDING FRICTION

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A one-dimensional elastic system with distributed contact under fixed boundary conditions is investigated in order to study dynamic behavior under sliding friction. A partial differential equation of motion is established and its exact solution is presented. Due to the friction the eigenvalue problem is non-self-adjoint. Mathematical methods for handling the non-self-adjoint system, such as the non-self-adjoint eigenvalue problem and the eigenvalue problem with a proper inner product, are reviewed and applied. The exact solution showed that the undamped elastic system under fixed boundary conditions is neutrally stable when the coefficient of friction is a constant. The assumed mode approximation and the lumped-parameter discretization method are evaluated and their solutions are compared with the exact solution. As a cautionary example the assumed modes approximation leads to false conclusions about stability. The lumped-parameter discretization algorithm generates reliable results.

1. INTRODUCTION

Friction-induced vibrations and the accompanying noise are serious problems in many industrial applications, for example brake systems in automobiles, rail–wheel systems, and machine–tool/work–piece systems in manufacturing. These various forms of vibrations are often undesirable not only because of their detrimental effects on the performance of the mechanical systems, but also as a source of discomfort in operating environments. In most of the previous research related to friction-induced vibrations, low-degree-of-freedom models have been used in order to explain dynamic stability of friction sliding and stick-slip vibrations. Despite its simplicity in modelling and analysis, such a system may have limitations in showing characteristic features of an elastic medium subject to distributed friction. Especially, in order to investigate a continuum in contact with a large area, a proper continuous model which can capture dynamic features and its mathematical method in handling the distributed frictional contacts is required.

In this paper, a mathematical model for a one-dimensional elastic material subjected to distributed friction contact is established and its dynamic stability is evaluated. Several discretization methods are evaluated. Assumed-mode methods lead to the general eigenvalue problem, which includes the non-self-adjoint eigenvalue problem and an eigenvalue problem using a proper inner product that transforms a non-self-adjoint problem to a self-adjoint problem. A cautionary example in applying the Galerkin's discretization method in the low order non-self-adjoint system is revealed. As an alternative

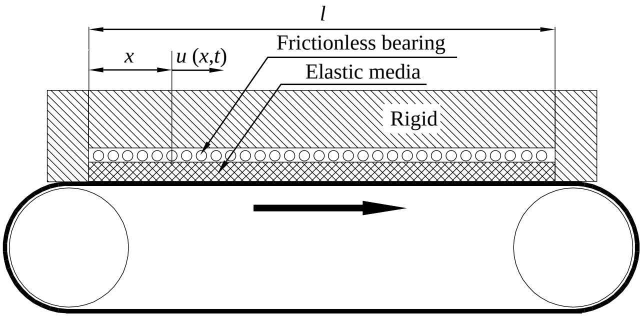

Figure 1. A schematic diagram for a one-dimensional elastic medium subjected to distributed friction. A medium is under fixed end boundary conditions. A frictionless linear bearing is installed on top of a medium so as to allow axial motions of an elastic medium.

discretization method a lumped-parameter method is applied and shows its validity in applications.

2. EQUATION OF MOTION

In the system shown in Figure 1, a linear elastic medium, placed between a moving belt and a frictionless linear bearing, represents a one-dimensional, undamped, continuous system in distributed sliding contact. Although a non-constant coefficient of friction has been known to be one of the crucial factors for system stability, the friction coefficient is assumed to be a constant with respect to relative speed. This situation is worth studying as it has been shown to be unstable in semi-infinite media [1, 2] and periodic boundary conditions [3, 4]. In addition, any parameters having random properties, such as roughness of contact surface, are not included in order to focus on the effects of uniform properties of materials. Moreover, non-uniform motions, such as stick-slip motion or loss of contacts, are not included.

A system composed of a linear elastic medium undergoes axial sliding. The equation of axial motion for a homogeneous, undamped elastic medium is

$$A\frac{\partial \sigma_x(x,t)}{\partial x} + f(x,t) = \rho\frac{\partial^2 u}{\partial t^2}, \tag{1}$$

where $A = wh$ is the cross-sectional area of an elastic medium of constant width w (into the page) and height h, ρ is the mass per unit length of the elastic material, $\sigma_x(x,t)$ is the stress over the cross-section, $u(x,t)$ is the axial displacement, and $f(x,t)$ is the friction force per unit length. Applying a linear stress–strain relationship, stress is expressed as $\sigma_x(x,t) = E\varepsilon_x(x,t)$, where E is Young's modulus of the material.

Axial stress accompanies a change in cross-sectional area in an open rod, due to the Poisson effect. Since this system is constrained, there is instead a change in normal stress, σ_y. Under plane stress, the friction force per unit length due to Poisson's effect is given by

$$f(x,t) = -\mu w\sigma_y(x,t) = -\mu w\{\sigma_0 + \nu\sigma_x(x,t)\}, \tag{2}$$

where μ is a friction coefficient, $\sigma_y(x,t)$ is a contact normal stress, and σ_0 is a pre-loaded normal stress, which should be always less than zero (compression) to generate friction force

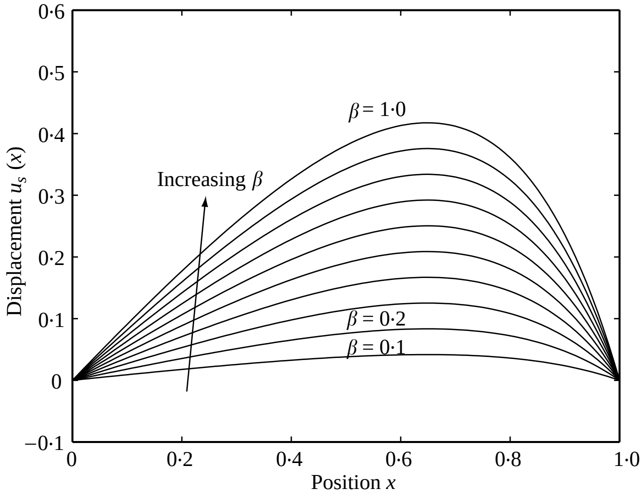

Figure 2. The exact static solution $u_s(x)$ by changing β in the one-dimensional system. Here β is in the range of 0·1–1·0 with increments of 0·1. In this example $\alpha = 4{\cdot}0$.

and maintain contact with the sliding rigid body. This distributed friction force contributes to the axial stresses in the medium through equation (1).

By considering the linear strain-displacement relation, $\varepsilon_x(x,t) = \partial u(x,t)/\partial x$, a non-dimensional equation of motion is obtained as

$$\frac{\partial^2 u^*}{\partial x^{*^2}} - \alpha \frac{\partial u^*}{\partial x^*} + \alpha\beta = \frac{\partial^2 u^*}{\partial t^{*^2}}. \tag{3}$$

The dimensionless parameters used in equation (3) are $\alpha = \mu w v l/A = \mu v l/h$, $\beta = -\sigma_0/vE$, $u^* = u/l, x^* = x/l$ and $t^* = t/\sqrt{(\rho l^2/AE)}$, where l denotes contact length, and u^*, x^* and t^* are the dimensionless displacement, co-ordinate and time respectively. For simplicity, the notation $*$ will be neglected in the following development.

For a typical system subjected to a distributed friction contact a fixed boundary condition is selected, such that

$$u(0,t) = u(1,t) = 0. \tag{4}$$

3. EXACT SOLUTION

A good way to handle the constant term in equation (3) is to separate the displacement into static and dynamic components:

$$u(x,t) = u_s(x) + u_d(x,t). \tag{5}$$

As such, we find

$$u_s(x) = -\frac{\beta l}{(e^{\alpha l} - 1)}(e^{\alpha l x} - 1) + \beta l x. \tag{6}$$

Figure 2 depicts the variation in the static solution $u_s(x)$ for β in the range of 0·1–1·0 with increments of 0·1 under the condition of $\alpha = 4{\cdot}0$. As β increases, i.e., as normal loads

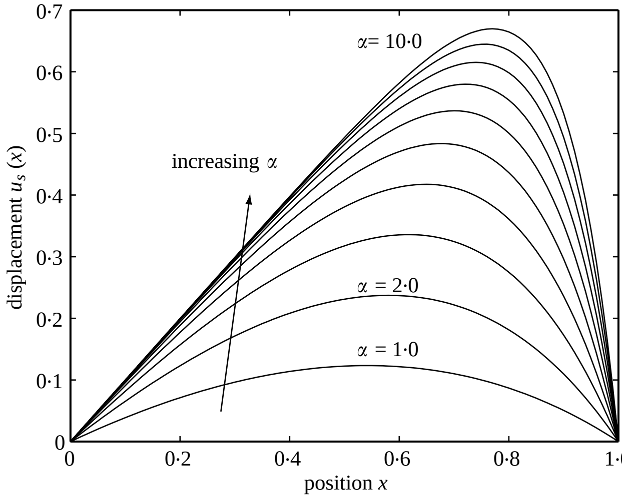

Figure 3. The exact static solution $u_s(x)$ by changing α in the one-dimensional system. Here α is in the range of 1·0–10·0 with increments of 1·0. In this example $\beta = 1{\cdot}0$.

increase, or Young's modulus decreases, the non-symmetric static solution along the x-axis gets larger. Figure 3 provides the trends of static solutions under variations in α from 1·0 to 10·0 with increments 1·0 with $\beta = 1{\cdot}0$; α influences the asymmetry of $u_s(x)$.

Then the term $\alpha\beta$ in equation (3) is eliminated by static solution $u_s(x)$ in equation (6), and a dynamic equation of motion in terms of $u_d(x, t)$ is obtained as

$$\frac{\partial^2 u_d}{\partial x^2} - \alpha \frac{\partial u_d}{\partial x} = \frac{\partial^2 u_d}{\partial t^2}. \tag{7}$$

The exact solution for the dynamic component of equation (7) satisfying the boundary condition (4) is then obtained by using the separation of variables method. Thus,

$$u_d(x, t) = \sum_{j=1}^{\infty} \sqrt{2}\mathrm{e}^{(\alpha/2)x} \sin(j\pi x)\,\{a_j \cos(\omega_j t) + b_j \sin(\omega_j t)\}, \tag{8}$$

where natural frequencies are $\omega_j = \sqrt{(j\pi)^2 + \alpha^2/4}$, and a_j, b_j are constants determined by initial conditions. The system is neutrally stable.

The first three exact mode shapes, which depend on parameter α in equation (8), are shown in Figure 4. Increasing α influences the shapes of the unsymmetric free-vibration eigenfunctions. However, it does not destabilize the system. In other words, α affects the mode shapes, which are non-symmetric along the x-axis, and α affects the natural frequencies in equation (8). Variations in α do not destabilize the dynamic system under fixed boundary conditions with a constant coefficient of friction.

4. SELF-ADJOINT AND NON-SELF-ADJOINT SYSTEMS

Numerous systems encountered in structural dynamics are self-adjoint with distinct eigenvalues. This means that such systems have symmetric properties. A self-adjoint system has real eigenvalues and eigenfunctions. Moreover, the eigenfunctions are orthogonal to

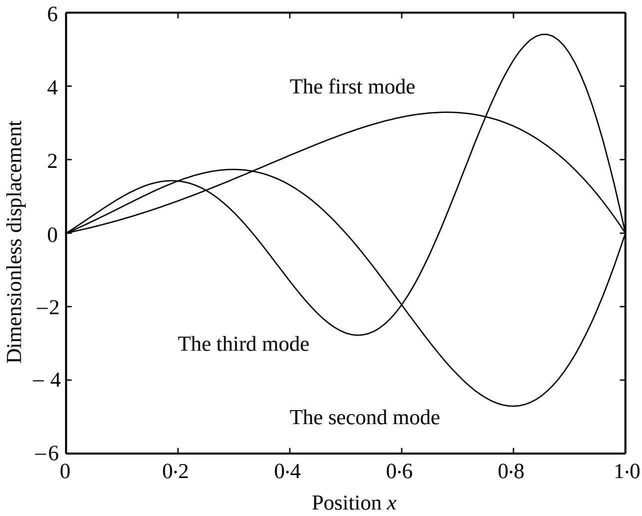

Figure 4. The first three eigenfunctions in the dynamic solution. ($\phi_j(x) = e^{(\alpha/2)x}\sin(j\pi x)$, where $j = 1, 2, 3$ with $\alpha = 4{\cdot}0$.)

each other. However, structural systems which endure aerodynamic forces, friction forces, and follower forces may lose their symmetries and become non-self-adjoint [5–9]. The orthogonal relations and the expansion theorem which have been developed on the bases of self-adjoint properties are no longer applicable to the non-self-adjoint systems.

Many non-self-adjoint systems can be transformed to self-adjoint systems by defining a proper inner product, so that the problem of non-self-adjointedness can ultimately be handled through similar procedures of the self-adjoint cases [10, 11]. (The techniques associated with transformations of system properties are presented in the next section.)

Let us consider our eigenvalue problem and the issue of self adjointness. Suppose the dynamic solution of equation (7) is represented in the form, $u_d(x,t) = \Phi(x)Q(t)$. Then the eigenvalue problem is given by

$$L\Phi = \lambda\Phi, \tag{9}$$

where the linear operator in equation (9) is defined by

$$L \stackrel{\text{def}}{=} -\frac{d^2}{dx^2} + \alpha\frac{d}{dx}, \tag{10}$$

with the boundary conditions of

$$\Phi(0) = \Phi(1) = 0. \tag{11}$$

We introduce the classical definition of an inner product between two functions as

$$\langle f, g\rangle \stackrel{\text{def}}{=} \int_D f(x)g(x)\,dx, \tag{12}$$

where D is the domain of the eigenvalue problem. Then, the operator L has an adjoint operator L^* defined by

$$\langle \Psi, L\Phi\rangle = \langle L^*\Psi, \Phi\rangle. \tag{13}$$

And the original system and its adjoint system can be written as

$$L\phi_i = \lambda_i \phi_i, \qquad L^* \psi_j = \lambda_j^* \psi_j, \tag{14a, b}$$

where λ_i and λ_j^* are real or complex eigenvalues corresponding to L and L^* respectively. The operator L^* is called the adjoint operator of L and the set of eigenfunctions ψ_j ($j = 1, 2, \dots$) is said to be adjoint to the set of eigenfunctions ϕ_i ($i = 1, 2, \dots$) over the defined classical inner product (12).

A large class of structural dynamic systems with conservative forces are *self-adjoint*, which means that $L = L^*$, and the two sets of eigenfunctions are the same for the corresponding eigenvalues. In such a case orthogonality is expressed as

$$\langle \phi_i, \phi_j \rangle = \int_D \phi_i \phi_j \, \mathrm{d}x = 0, \quad i \neq j, \qquad i, j = 1, 2, \dots, \infty. \tag{15}$$

By using the orthogonality, coefficients of any function $w(x, t) = \sum_{j=1}^{\infty} \phi_j(x) q_j(t)$ can be written as

$$q_i = \langle \phi_i, w \rangle = \left\langle \phi_i, \sum_{j=1}^{\infty} \phi_j q_j \right\rangle. \tag{16}$$

This is called the *expansion theorem* for self-adjoint systems.

However, if $L \neq L^*$, the system is non-self-adjoint, and the orthogonality in equation (15) does not hold. For the case in which $L \neq L^*$, multiplying equation (14a) by ψ_j, and equation (14b) by ϕ_i, and then integrating over the domain D yields

$$\langle \psi_j, L\phi_i \rangle = \int_D \psi_j L \phi_i \, \mathrm{d}x = \lambda_i \int_D \psi_j \phi_i \, \mathrm{d}x, \tag{17a}$$

$$\langle \phi_i, L^* \psi_j \rangle = \int_D \phi_i L^* \psi_j \, \mathrm{d}x = \lambda_j^* \int_D \phi_i \psi_j \, \mathrm{d}x. \tag{17b}$$

Subtracting equations (17) leads to

$$(\lambda_i - \lambda_j^*) \int_D \phi_i \psi_j \, \mathrm{d}x = 0. \tag{18}$$

Hence, if $\lambda_i \neq \lambda_j^*$

$$\langle \phi_i, \psi_j \rangle = \int_D \phi_i \psi_j \, \mathrm{d}x = 0, \quad i \neq j, \quad i, j = 1, 2, \dots, \infty. \tag{19}$$

This is the *biorthogonality* of eigenfunctions ϕ_i and ψ_j, which means an eigenfunction of L corresponding to an eigenvalue λ_i is orthogonal to an eigenfunction of L^* corresponding to λ_j^*, where the λ_i is distinct from λ_j^*. The non-self-adjoint operator L has the same eigenvalues as the operator L^*. The general expansion theorem related to non-self-adjoint systems, called the dual-expansion theorem, is presented in the works by Meirovitch [5] and MacCluer [10].

Let us return to the problem of interest. In order to seek the adjoint operator L^* of this study, we examine the adjoint operator L^* defined in equation (13):

$$\int_0^1 \Psi L \Phi \, \mathrm{d}x = \int_0^1 \Psi \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \alpha \frac{\mathrm{d}}{\mathrm{d}x} \right) \Phi \, \mathrm{d}x$$

$$= \int_0^1 \Phi \left(-\frac{\mathrm{d}^2 \Psi}{\mathrm{d}x^2} - \alpha \frac{\mathrm{d}\Psi}{\mathrm{d}x} \right) \mathrm{d}x$$

$$= \int_0^1 \Phi L^* \Psi \, \mathrm{d}x, \tag{20}$$

where the boundary conditions of equation (4) have been accounted for. Thus, the adjoint operator of this study is

$$L^* \stackrel{\mathrm{def}}{=} -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \alpha \frac{\mathrm{d}}{\mathrm{d}x} \tag{21}$$

with zero boundary conditions. The adjoint operator L^* in equation (21) is different from the operator L in equation (10). Assuming that dynamic solutions of this study can be represented by

$$u_d(x, t) = \sum_{j=1}^{\infty} \phi_j(x) \, q_j(t), \tag{22}$$

then, by multiplying adjoint eigenfunction ψ_i, and using the biorthogonality in equation (19), coefficients $q_j(t)$ are obtained as

$$q_j(t) = \langle \psi_i, u_d \rangle = \left\langle \psi_i, \sum_{j=1}^{\infty} \phi_j(x) \, q_j(t) \right\rangle. \tag{23}$$

Here, the eigenfunctions are

$$\psi_i(x) = \sqrt{2} \mathrm{e}^{-(\alpha/2)x} \sin(i\pi x), \qquad \phi_j(x) = \sqrt{2} \mathrm{e}^{(\alpha/2)x} \sin(j\pi x), \quad i, j = 1, 2, \ldots, \infty. \tag{24a, b}$$

By multiplying the normalized adjoint eigenfunction, $\psi_i(x)$, with equation (9) and integrating from 0 to 1, an infinite set of decoupled ordinary differential equations is obtained as

$$\sum_{j=1}^{\infty} m_{ij} \ddot{q}_j + \sum_{j=1}^{\infty} k_{ij} q_j = 0, \quad i = 1, 2, \ldots, \infty, \tag{25}$$

where

$$m_{ij} = \langle \psi_i, \phi_j \rangle = \int_0^1 \psi_i \phi_j \, \mathrm{d}x = \delta_{ij}, \tag{26a}$$

$$k_{ij} = \langle \psi_i, L\phi_j \rangle = \int_0^1 \psi_i L \phi_j \, \mathrm{d}x = \omega_j^2 \delta_{ij} = \{(j\pi)^2 + \alpha^2/4\} \delta_{ij}, \quad i, j = 1, 2, \ldots, \infty. \tag{26b}$$

Consequently, the projection by using the adjoint eigenfunctions in the non-self-adjoint system yields the set of decoupled ordinary differential equations. In addition, it is verified that eigenvalues derived from general eigenvalue problems are the same as the exact solutions derived in the previous section.

5. EIGENVALUE PROBLEM BASED ON A PROPER INNER PRODUCT

The eigenfunctions derived in the previous section are not mutually orthogonal since the system has a non-self-adjoint operator. However, the "folklore" is that a non-self-adjoint problem can be cast as self-adjoint by using a proper inner product when the problem possesses a meaningful discrete spectral structure [10]. In this section, the method for choosing a proper inner product, or equivalently recasting the form of the partial differential equation which enables the system to be self-adjoint, is reviewed. Then this method is applied to the problem of interest in order to suggest an alternative way in solving the general eigenvalue problem.

The general second order partial differential equation in the form of

$$p_0(x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_2(x)y + \lambda p_3(x)y = 0, \tag{27}$$

with the auxiliary homogeneous boundary conditions

$$a_0 y(x_0) + a_1\frac{\mathrm{d}y(x_0)}{\mathrm{d}x} + a_2 y(x_1) + a_3\frac{\mathrm{d}y(x_1)}{\mathrm{d}x} = 0, \tag{28a}$$

$$b_0 y(x_0) + b_1\frac{\mathrm{d}y(x_0)}{\mathrm{d}x} + b_2 y(x_1) + b_3\frac{\mathrm{d}y(x_1)}{\mathrm{d}x} = 0 \tag{28b}$$

is defined on the interval (x_0, x_1). This is the Sturm–Liouville problem subject to homogeneous boundary conditions [11, 12]. Suppose that the coefficients $p_0(x)$ and $p_3(x)$ are positive and the $p_0(x), p_1(x)$, and $p_3(x)$ are twice differentiable. Letting

$$p(x) = \mathrm{e}^{\int_{x_0}^{x}(p_1(x)/p_0(x))\,\mathrm{d}x}, \qquad q(x) = \frac{p_2(x)p(x)}{p_0(x)}, \qquad g(x) = \frac{p_3(x)p(x)}{p_0(x)} \tag{29}$$

and multiplying equation (27) by weighting function $p(x)/p_0(x)$, yields

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{p(x)\frac{\mathrm{d}y}{\mathrm{d}x}\right\} + \{q(x) + \lambda g(x)\}y = 0 \tag{30}$$

which is a more convenient self-adjoint form. Thus by multiplying equation (27) by the weight function $p(x)/p_0(x)$, the system is shown to be self-adjoint.

Returning to the problem of interest, the equation of motion (9) can be transformed to a self-adjoint system by using the weight function $\mathrm{e}^{-\alpha x}$.

Thus, the eigenvalue problem in self-adjoint form is given by

$$\tilde{L}\Phi = \lambda w(x)\Phi, \tag{31}$$

where

$$\tilde{L} \stackrel{\mathrm{def}}{=} -\frac{\mathrm{d}}{\mathrm{d}x}\left\{\mathrm{e}^{-\alpha x}\frac{\mathrm{d}}{\mathrm{d}x}\right\} \tag{32}$$

and the weight function is $w(x) = \mathrm{e}^{-\alpha x}$.

The self-adjointedness of operator $\tilde{L}$ is verified by taking the classical inner product (12) and integrating by parts, such that

$$\begin{aligned}\langle \Psi, \tilde{L}\Phi \rangle &= -\int_0^1 \Psi \frac{\mathrm{d}}{\mathrm{d}x}\left\{ \mathrm{e}^{-\alpha x}\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right\}\mathrm{d}x \\ &= \int_0^1 \mathrm{e}^{-\alpha x}\frac{\mathrm{d}\Phi}{\mathrm{d}x}\frac{\mathrm{d}\Psi}{\mathrm{d}x}\mathrm{d}x \\ &= \langle \Phi, \tilde{L}\Psi \rangle. \end{aligned} \tag{33}$$

In addition, the positive definiteness also can be shown from the fact that

$$\begin{aligned}\langle \Phi, \tilde{L}\Phi \rangle &= -\int_0^1 \Phi \frac{\mathrm{d}}{\mathrm{d}x}\left\{ \mathrm{e}^{-\alpha x}\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right\}\mathrm{d}x \\ &= \int_0^1 \mathrm{e}^{-\alpha x}\left\{\frac{\mathrm{d}\Phi}{\mathrm{d}x}\right\}^2 \mathrm{d}x \geqslant 0 \end{aligned} \tag{34}$$

is always non-negative. It is equal to zero only if $\Phi(x)$ is a constant throughout the domain. Because of the boundary condition (4), however, this constant must be zero, which would imply a trivial solution. It follows that the operator $\tilde{L}$ in equation (32) is positive definite. Therefore, the non-self-adjoint operator L described in equation (10) is transformed to the self-adjoint positive definite operator $\tilde{L}$ in equation (32) by applying the weight function $\mathrm{e}^{-\alpha x}$.

Identical results are also obtained by taking the weighted inner product, defined as

$$\langle f, g \rangle_w \overset{\mathrm{def}}{=} \int f(x)g(x)w(x)\,\mathrm{d}x, \tag{35}$$

where $w(x)$ is weight function. By choosing a weight function $w(x) = \mathrm{e}^{-\alpha x}$, we can verify the self-adjointedness with respect to the weighted inner product as

$$\langle \Phi, L\Psi \rangle_w = \langle \Psi, L\Phi \rangle_w, \tag{36}$$

where the operator L is defined in equation (10). Hence the terminology "proper inner product". An applied example can be found in Chait *et al.* [13].

The equation of motion (31) is identical to the equation of axial free motion for an elastic rod having varying stiffness $\mathrm{e}^{-\alpha x}$ and varying mass distribution $\mathrm{e}^{-\alpha x}$ without friction, which we refer to as an "exponential rod".

The discretized equation of motion can be presented by applying Lagrange's formula to the equivalent "exponential rod". Suppose that the solution $u_d(x, t)$ can be written as a series

$$u_d(x,t) = \sum_{j=1}^{\infty} \phi_j(x)\, r_j(t), \tag{37}$$

where $\phi_j(x)$ can be any admissible function without loss of generality. The kinetic and potential energies of a continuous system have integral expressions. The kinetic energy can be written in the familiar form of

$$T(t) = \frac{1}{2}\int_0^1 \mathrm{e}^{-\alpha x}\left\{\frac{\partial u_d(x,t)}{\partial t}\right\}^2 \mathrm{d}x. \tag{38}$$

In a similar expression, the potential energy can be written as

$$V(t) = \frac{1}{2}\int_0^1 \mathrm{e}^{-\alpha x}\left\{\frac{\partial u_d(x,t)}{\partial x}\right\}^2 \mathrm{d}x. \tag{39}$$

The natural boundary conditions are of no concern here because they are automatically accounted for in the kinetic and potential energies. Consider Lagrange's equations for conservative systems, namely,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{r}_j}\right) - \frac{\partial T}{\partial r_j} + \frac{\partial V}{\partial r_j} = 0, \quad j = 1, 2, \ldots, \infty. \tag{40}$$

The equation of motion in discretized form is obtained by

$$\sum_{j=1}^{\infty} m_{ij}\frac{\mathrm{d}^2 r_j(t)}{\mathrm{d}t^2} + \sum_{j=1}^{\infty} k_{ij} r_j(t) = 0, \tag{41}$$

where

$$m_{ij} = \int_0^1 \mathrm{e}^{-\alpha x}\phi_i(x)\phi_j(x)\,\mathrm{d}x, \qquad k_{ij} = \int_0^1 \mathrm{e}^{-\alpha x}\frac{\mathrm{d}\phi_i(x)}{\mathrm{d}x}\frac{\mathrm{d}\phi_j(x)}{\mathrm{d}x}\,\mathrm{d}x, \quad i, j = 1, 2, \ldots, \infty. \tag{42a, b}$$

By selecting the set of $\phi_j(x)$ as normalized eigenfunction of equation (31) in equation (42), i.e., $\phi_j(x) = \sqrt{2}\mathrm{e}^{(\alpha/2)x}\sin(j\pi x)$ from the results of the previous section, the discretized uncoupled equations of motion are obtained. The eigenvalues for this discrete system, which are $\lambda_j = (j\pi)^2 + \alpha^2/4$, are identical to the exact solution (8).

Equations (38) and (39) can be thought of in terms of the original friction system as pseudo-energies defined via the weighted inner product. The friction and Poisson's effects work into the pseudo-energies through the $\mathrm{e}^{-\alpha x}$ terms, resembling "effective" mass and stiffness distributions. As such, the generalized forces normally needed in Lagrange's equations are accounted for in equation (40).

Thus, it is verified that the system having non-orthogonality in its eigenfunctions is a minor matter, and it is correctable by projecting under the proper inner product.

6. NON-CONVERGENCE OF GALERKIN'S METHOD: A CAUTIONARY EXAMPLE

The exact solution from section 3 shows that this system's dynamic stability is not dependent on the system parameters. The system is neutrally stable, behaving like an undamped vibration system with natural frequencies of $\omega_j = \sqrt{(j\pi)^2 + \alpha^2/4}$. A change in β changes the system's static solution and has no influence on the linear stability. With the addition of modal damping, the eigenvalues will have negative real parts and steady sliding is expected to be asymptotically stable.

In this section, the assumed mode projection—Galerkin's projection—is applied in the evaluation of system stability in order to verify the feasibility of applying an approximate method. Even though the exact eigenvalue solutions have been obtained already in the previous sections, the application of an approximate discretization method may provide a cautionary example for its use.

We apply the assumed-mode method to approximate the non-self-adjoint equation (3) as a set of ordinary differential equations. The dynamic response $u_d(x, t)$ can be represented with assumed modes satisfying the geometric boundary conditions and p derivatives in the partial differential equation of order $2p$, where $p = 1$. Here,

$$u_d(x,t) = \sum_{j=1}^{\infty} \hat{\phi}_j(x) a_j(t), \tag{43}$$

where $\hat{\phi}_j(x) = \sqrt{2}\sin(j\pi x)$ is chosen as an approximate mode. After projecting with these assumed modes, an approximate ordinary different equation of motion is

$$\sum_{j=1}^{\infty} m_{ij} \frac{\mathrm{d}^2 a_j}{\mathrm{d}t^2} + \sum_{j=1}^{\infty} k_{ij} a_j = f_i, \quad i = 1, 2, \ldots \infty, \tag{44}$$

where

$$m_{ij} = \delta_{ij}, \qquad k_{ij} = k_{ij}^S + k_{ij}^A, \tag{45a}$$

$$k_{ij}^S = (j\pi)^2 \delta_{ij} \quad k_{ij}^A = \begin{cases} \dfrac{4\alpha ij}{i^2 - j^2}, & |i-j| = \text{odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{45b}$$

$$f_i = \begin{cases} \dfrac{2\sqrt{2}\alpha\beta}{i\pi}, & i = \text{ odd}, \\ 0, & \text{otherwise}, \end{cases} \tag{45c}$$

where k^S and k^A are symmetric and antisymmetric stiffness matrices respectively.

Focusing on the low-dimensional dynamics, the system can be approximated with n coupled ordinary differential equations. The real parts of the eigenvalues of this system indicate predicted stability characteristics. The dependency of eigenvalues on parameters by including two assumed modes is shown in Figure 5. Instability apparently occurs when the real part of an eigenvalue is positive at the critical condition $\alpha = 5{\cdot}7$, accompanied by a collision between two frequencies. This instability mechanism resembles flutter, and has been seen as one of possible instability mechanisms, e.g., flow-induced vibrations [14] and friction-induced vibrations [15].

However, these results contradict the exact solution since it has no instability mechanism involving parameter α, based on the results of section 3. Bolotin [14] had investigated this "paradox" for flow across a membrane. The work showed non-convergent characteristics in the assumed mode projections, and gave a theoretical criterion for convergence based on the linear operator. According to those results, conservative systems with second order operators are not guaranteed to converge in assumed-mode approximations.

Non-convergence of this eigenvalue problem can be demonstrated by increasing the number of assumed modes. Figures 6 and 7 show the imaginary and real part eigenvalues for 3–5 modes respectively. Considering Figure 5 also, the two lowest-frequency modes interact at $\alpha = 5{\cdot}7$, and $\alpha = 9{\cdot}0$ for two- and four-mode approximations, but do not interact for three- and five-mode approximations. This shows that the approximated solution by using assumed mode methods for finding the smallest interaction value α fails to converge

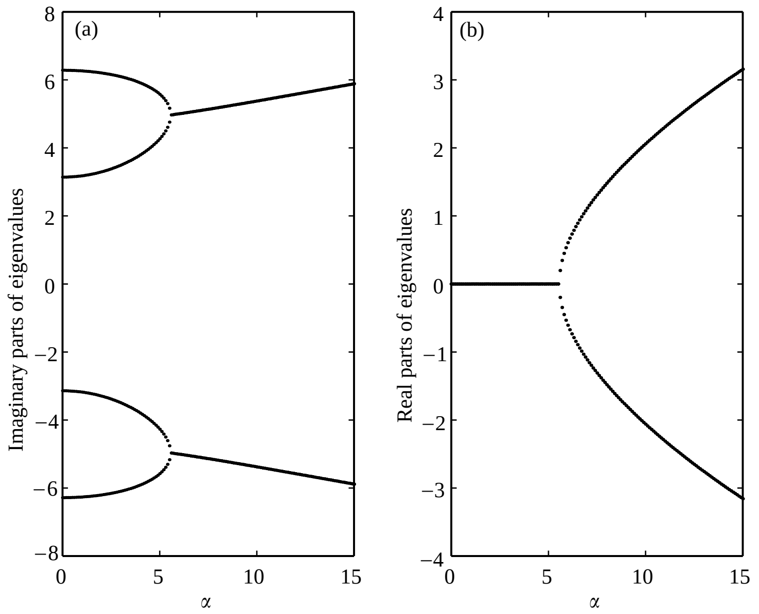

Figure 5. A non-convergent result: eigenvalues versus α in the one-dimensional friction system by applying the assumed mode method with two modes included. (a) Imaginary and (b) real parts of the eigenvalues versus α are shown. The selected assumed modes are $\hat{\phi}_j(x) = \sqrt{2}\sin(j\pi x)$ for $j = 1, 2$.

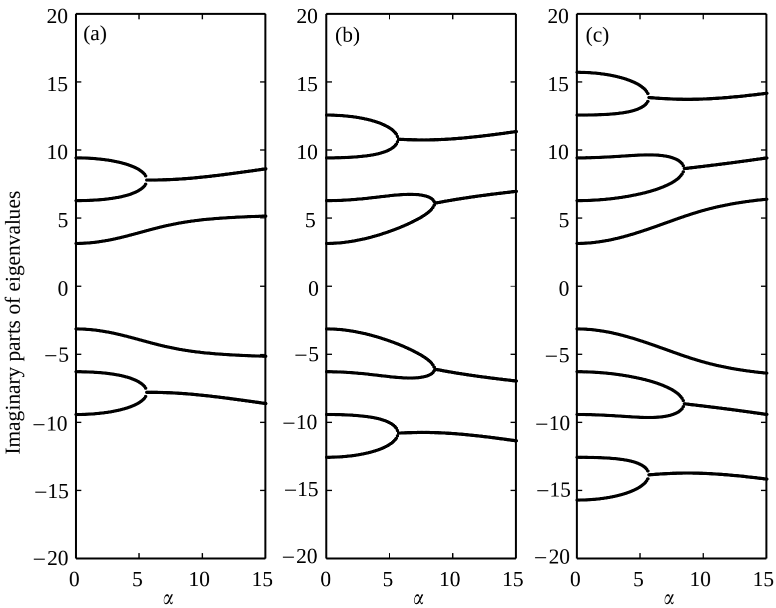

Figure 6. A non-convergent result: imaginary parts of the eigenvalues versus α by applying the assumed mode method in the one-dimensional friction system: (a) three modes, (b) four modes, and (c) five modes are included. The selected assumed modes are $\hat{\phi}_j(x) = \sqrt{2}\sin(j\pi x)$.

with an increasing number of modal co-ordinates. This hints at faulty results when applying the assumed-mode method to this problem. Failure to recognize the poor result can lead to false conclusions about system stability.

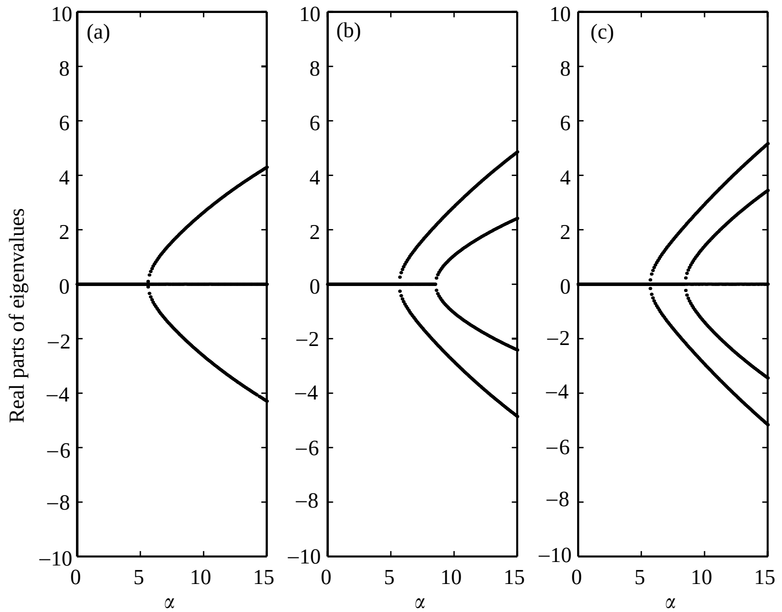

Figure 7. A non-convergent result: real parts of the eigenvalues versus α by applying the assumed mode method in the one-dimensional friction system: (a) three modes, (b) four modes, and (c) five modes are included. The selected assumed modes are $\hat{\phi}_j(x) = \sqrt{2}\sin(j\pi x)$.

The conditions for non-convergence have been shown by checking the matrix determinant by Bolotin [14]. Consider the convergence of the determinant in equation (44). Equation (44) can be written as

$$\frac{\mathrm{d}^2 a_i}{\mathrm{d}t^2} + \Omega_i^2 a_i + \eta \sum_{j=1}^{\infty} b_{ij} a_j = 0, \quad i = 1, 2, \ldots, \infty. \tag{46}$$

And the characteristic determinant becomes

$$\Delta = |(\Omega_i^2 - \lambda)\delta_{ij} + \eta b_{ij}| = 0. \tag{47}$$

Dividing the ith row by Ω_i and the jth column by Ω_j, the determinant Δ can be expressed in the form

$$\Delta = |\delta_{ij} + c_{ij}|. \tag{48}$$

According to Bolotin [14] and Kantorovich and Krylov [16], the infinite determinant converges if the double series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |c_{ij}| \tag{49}$$

converges. The determinant is described as normal when condition (49) converges. By checking the determinant of equation (44), it diverges with an infinite number of modes. Thus it is not a normal determinant.

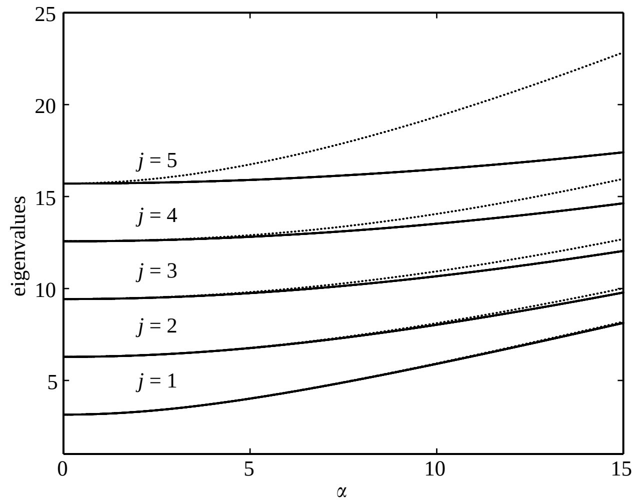

Figure 8. Comparison between the exact and approximate eigenvalues: the square roots of the exact eigenvalues, $\sqrt{\lambda_j} = \sqrt{(j\pi)^2 + \alpha^2/4}$, are shown with the solid lines. The mode-projected approximate eigenvalues obtained from a self-adjoint system are shown with dotted line. The selected assumed modes are $\hat{\phi}_j(x) = \sqrt{2}\sin(j\pi x)$ for $j = 1, 2, 3, 4, 5$.

Figure 8 presents the exact and approximate eigenvalues based on the proper inner product versus α with five assumed modes selected. The low-frequency approximation is accurate in the eigenvalue calculation. Though there are still slight deviations from the exact solution in high-frequency eigenvalue approximations, a more accurate approximation is expected by including more modes. Consequently, a false indication of instability has been avoided in evaluating the eigenvalues for the self-adjoint representation of the system.

There do exist investigations into the approximation of non-self-adjoint systems. Meirovitch and Hagedorn [17] investigated the modelling of distributed non-self-adjoint systems, such as damped boundary condition models. In using the method of weighted residuals to produce the approximate solution to the eigenvalue problem, the displacement of a non-self-adjoint system is ordinarily represented by a linear combination of comparison functions, i.e., functions that satisfy all the boundary conditions. Because of difficulties in finding comparison functions, a more feasible approach involves the construction of an approximate solution by using combinations of admissible functions, called quasi-comparison functions, capable of satisfying all the geometric boundary conditions of the problem [17]. Similar approaches for solving the approximate solutions can be found in Meirovitch and Kwak [18], and Hagedorn [19]. The proof of Galerkin's method for non-self-adjoint boundary value problems has been given by Diprima and Sani [20] and sensitivity analyses in the non-conservative problem by using adjoint variational method are presented by Prasad and Herrmann [21], and Pedersen and Seyranian [22].

7. A LUMPED-PARAMETER DISCRETIZATION

Since the classical approximation method which relies on the modal co-ordinates may not be valid, and the convergence of eigenvalues not guaranteed, there is no reason to expect other discretizations to converge, either. But we investigate the performance of other

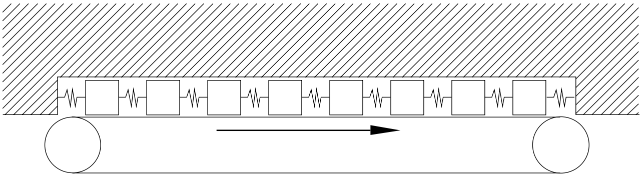

Figure 9. A schematic diagram for the undamped, lumped-parameter model subjected to distributed friction. Fixed end boundary conditions are applied.

discretizations in the hope that those difficulties are overcome, so that the discretization can be applied later to non-linear studies with some confidence.

Consider the system shown in Figure 9, which shows the lumped-parameter model from the continuous system in Figure 1. The mass blocks connected to linear springs are placed on the moving belt. There are frictional forces between the mass blocks and the moving belt. In this model, each mass block represents not only a lumped mass, but also a discrete elastic mass which can contract and elongate based on the Poisson effect due to the forces exerted around the mass. Since the normal expansion is restricted as shown in Figure 9, the contraction and elongation influence the normal load, which causes the variation of the friction forces. The equation of motion for undamped ith mass is written as

$$m\ddot{x}_i(t) + k\{-x_{i-1}(t) + 2x_i(t) - x_{i+1}(t)\} + f_i(t) = 0, \tag{50}$$

where m is the mass of each mass block, k is the spring stiffness, $x_i(t)$, $\dot{x}_i(t)$, and $\ddot{x}_i(t)$ represent the displacement, velocity and acceleration of the ith mass, respectively, and $f_i(t)$ is the friction forces on the ith mass. Here the mass and stiffness are lumped from the evenly distributed system. Let us include Poisson ratio effect. Then the friction force is

$$f_i(t) = \mu N_i(t) = \mu[N_0 + vk\{x_i(t) - x_{i-1}(t)\}], \tag{51}$$

where μ is the friction coefficient and N_0 is the normal load on each block, which is a negative constant value ($N_0 < 0$), and $N_i(t)$ is the resultant normal load including Poisson ratio effects ($N_i(t) < 0$). Thus, the undamped equation of motion for the ith mass block is

$$\ddot{x}_i(\tau) - (1 + \mu v)x_{i-1}(\tau) + (2 + v\mu)x_i(\tau) - x_{i+1}(\tau) + \mu N_0/k = 0, \tag{52}$$

where $\tau = \omega_p t$, $\omega_p^2 = k/m$ and the time derivative $(\cdot)$ denotes $\partial/\partial\tau$. This is a difference equation of motion of the continuous system in equation (3).

Consider the fixed boundary condition of

$$x_0(t) = x_{n+1}(t) = 0, \qquad \frac{\mathrm{d}x_0(t)}{\mathrm{d}t} = \frac{\mathrm{d}x_{n+1}(t)}{\mathrm{d}t} = 0. \tag{53a, b}$$

The equation of motion for the undamped system is expressed by

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{f}_0, \tag{54}$$

where

$$\mathbf{M} = \mathbf{I}, \qquad \mathbf{K} = \begin{bmatrix} 2+\nu\mu & -1 & 0 & \dots & 0 & 0 \\ -(1+\nu\mu) & 2+\nu\mu & -1 & \dots & 0 & 0 \\ 0 & -(1+\nu\mu) & 2+\nu\mu & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -(1+\nu\mu) & 2+\nu\mu \end{bmatrix}, \tag{55a, b}$$

$$\mathbf{f}_0 = -\mu N_0/k\,[1, 1, \dots, 1]^{\mathrm{T}}. \tag{55c}$$

The matrix **I** denotes an identity matrix. The stiffness matrix **K** is non-symmetric due to the effect of friction and Poisson's ratio. The eigenvalues for the dynamic system are evaluated with respect to the static equilibrium. Thus, the eigenvalue problem is represented by

$$\lambda \mathbf{M} \boldsymbol{\phi} = \mathbf{K} \boldsymbol{\phi}. \tag{56}$$

Since $\mathbf{K} \neq \mathbf{K}^{\mathrm{T}}$, the orthogonality relations obtained from the symmetric properties are no longer valid. Furthermore, the expansion theorem derived from the symmetric relations can-not be applied to decompose any arbitrary vectors in terms of a set of eigenvectors.

Let us briefly review the general eigenvalue problem, which covers the non-symmetric properties in equation (56), and then return to the problem of interest. The transposed eigenvalue problem associated with equation (56) has the form

$$\lambda \mathbf{M} \boldsymbol{\psi} = \mathbf{K}^{\mathrm{T}} \boldsymbol{\psi}. \tag{57}$$

The eigenvalues of equation (57) are the same as those of equation (56). On the other hand, the eigenvectors of equation (57) are different from those of equation (56). Consider two distinct solutions of equations (56) and (57). These solutions satisfy the equations

$$\lambda_i \mathbf{M} \phi_i = \mathbf{K} \phi_i, \quad i = 1, 2, \dots, n \tag{58}$$

and

$$\lambda_j \mathbf{M} \psi_j = \mathbf{K}^{\mathrm{T}} \psi_j, \quad j = 1, 2, \dots, n. \tag{59}$$

Equation (59) can also be written in the left eigenvector form as

$$\lambda_j \psi_j^{\mathrm{T}} \mathbf{M} = \psi_j^{\mathrm{T}} \mathbf{K}, \quad j = 1, 2, \dots, n. \tag{60}$$

Multiplying equation (58) on the left by ψ_i^{T} and equation (60) on the right by ϕ_i and subtracting one result from the other, one obtains

$$(\lambda_i - \lambda_j)\, \psi_j^{\mathrm{T}} \phi_i = 0, \tag{61}$$

so that for distinct eigenvalues

$$\psi_j^{\mathrm{T}} \phi_i = 0, \quad \lambda_i \neq \lambda_j, \qquad i, j = 1, 2, \dots, n. \tag{62}$$

This means that the left eigenvectors and right eigenvectors of the system corresponding to distinct eigenvalues are orthogonal. It should be stressed that the eigenvectors are not

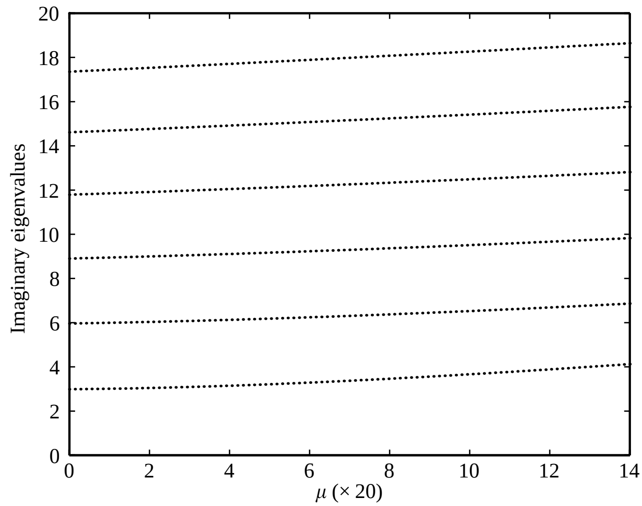

Figure 10. Trajectories of the eigenvalues versus friction coefficient μ in the undamped, lumped-parameter model.

mutually orthogonal in the same ordinary sense as those associated with the Hermitian matrix. Indeed, the two sets of eigenvectors ϕ_i and ψ_j are biorthogonal.

The fact that the eigenvectors ϕ_i and ψ_j are biorthogonal enables an expansion theorem for the general case. Assuming that any vector can be represented by an infinite sum of eigenvectors, there is a choice of expanding any arbitrary n-vector $\mathbf{x}$ in terms of the eigenvectors ϕ_i or ψ_j. For example,

$$\mathbf{x} = \mathbf{\Phi q}, \tag{63}$$

where $\mathbf{q} = [q_1, q_2, \dots, q_n]^{\mathrm{T}}$ is the vector of associated coefficients and $\mathbf{\Phi}$ is a modal matrix. Thus, the coefficients are obtained by

$$\mathbf{q} = \mathbf{\Psi}^{\mathrm{T}}\mathbf{x}, \tag{64}$$

where $\mathbf{\Psi}$ is the adjoint modal matrix.

Similarly, an expansion in terms of the eigenvector ψ_j has the form

$$\mathbf{x} = \mathbf{\Psi r}, \qquad \mathbf{r} = \mathbf{\Phi}^{\mathrm{T}}\mathbf{x}, \tag{65}$$

where $\mathbf{r} = [r_1, r_2, \dots, r_n]^{\mathrm{T}}$ is the vector of coefficients associated with ψ_j. This procedure, which treats the non-symmetric eigenvalue problem in the lumped-parameter system, corresponds to the non-self-adjoint eigenvalue problem in the continuous system discussed in the previous sections.

Let us return to the problem of interest. Figure 10 shows the numerical results of the eigenvalues by changing the friction coefficient μ, which is assumed to be constant with respect to the relative speed. The calculated eigenvalues can also be compared to the exact eigenvalues of the continuous system, shown in Figure 8.

As μ increases the frequencies simply increase and no destabilizations are found in Figure 10. This result shows a close approximation to the exact eigenvalues of the continuous system (Figure 8). Compared to the exact eigenvalues, the eigenvalues obtained from the lumped-parameter model are usually underestimated. There are no contradictory

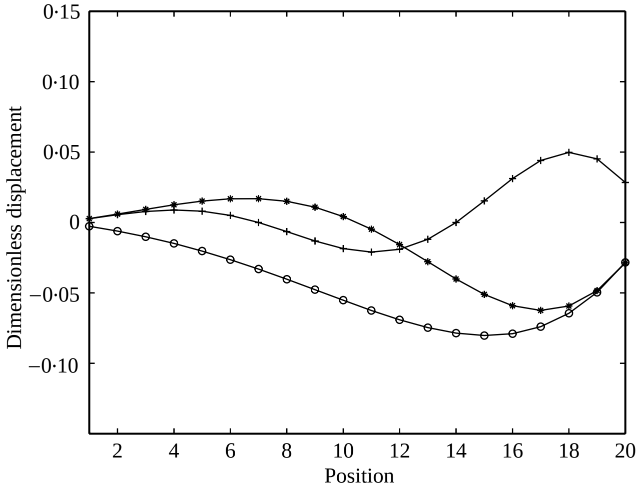

Figure 11. The non-symmetric eigenvectors corresponding to the three lowest eigenvalues: ○, first eigenvector; *, second eigenvector; +, third eigenvector.

results in evaluating the approximate eigenvalues in the lumped-parameter model since the numerical method used in this study (MATLAB) utilizes an adjoint property in evaluating the eigenvalues. We can expect a numerical analysis using such an algorithm to generate reliable results.

Figure 11 shows eigenvectors corresponding to the three lowest frequencies. These results show a close approximation to the exact eigenfunctions obtained from the continuous model in Figure 4.

8. CONCLUSION

The discretization of a one-dimensional continuous system with distributed sliding contact under fixed boundary conditions and a constant coefficient of friction was investigated. A partial differential equation of motion was established and its exact solution was presented. The exact solution shows that the undamped elastic system under fixed boundary conditions is neutrally stable when the coefficient of friction is a constant. An eigenvalue problem in this non-self-adjoint system was shown and its solution was provided with approaches based on both the non-self-adjoint eigenvalue problem and the eigenvalue problem with a proper inner product. A technique for choosing a proper inner product which reveals self-adjointedness was reviewed as well. Difficulties in evaluating the approximate eigenvalues can be overcome with the help of the proper inner product.

A contradictory result between the exact solution and the assumed-modes approximation in evaluating the eigenvalues was shown as a cautionary example. In this case, non-convergence of the assumed modes method can be easily detected. Projection of assumed modes with the proper inner product led to close approximations of the exact eigenvalues.

The lumped parameter discretization method generates reliable estimates of eigenvalues, modal functions, and stability. So we can apply that discretization to further non-linear analysis with better confidence.

With thorough understanding of the proper discretization of this system, further work can proceed. Topics of future study include non-linear stick-slip responses, refinements of friction characteristics, and boundary conditions. Effects of non-linear friction characteristics and boundary conditions on the system stability remain as a topic of future study.

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